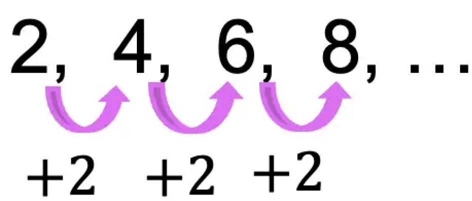
Grade 6 Math Circles
February 20-22, 2024
Arithmetic Sequences

What is a Sequence?

A sequence is an ordered list of numbers that often follows some pattern. Each number in a sequence is called a term, and a sequence can have any number of terms. One example of a sequence is

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64...$$

Here, we have a sequence of infinite (never-ending) numbers starting at 1. Every time we jump one number to the right, we multiply the previous number by two. We go from 1, to $1 \cdot 2 = 2$, to $2 \cdot 2 = 4$, to $4 \cdot 2 = 8$, and so forth. This pattern of multiplying or dividing each number to give us the next is called a *geometric sequence*. Another type of sequence would be to **add** a particular number at each step — this is called an *arithmetic sequence*, which we will be focusing on in this lesson. An example from mathsux of an arithmetic sequence is all positive even numbers:



Here, we have 4 terms of a sequence starting at 2, and we jump by a value of 2 each term we move to the right. This is called the *common difference* between each term. To find the common difference d, all we need to do is find any number after the first term in the sequence and subtract the previous number from it. For the sequence above, we can see that the 2^{nd} term is 4, and the term before it is 2. So the common difference would be:

$$d = 4 - 2 = 2$$

The starting value, often called a, is just the first term of the sequence, which is 2 in this case.

Example 1

Find the common difference d and starting value a for the arithmetic sequence

$$2,\ 5,\ 8,\ 11,\ 14...$$

Solution:

The starting value is 2, so $a = 2$. To find the common difference, we can subtract the 1^{st} term from the 2^{nd} to get $d = 5 - 2 = 3$. We could also subtract the 2^{nd} term from the 3^{rd}, etc.

Note that the difference can also be negative, but we find it the exact same way.

Exercise 1

Find the values of a and d for the following sequence:

$$3,\ -2,\ -7,\ -12,\ -17,\ -22$$

Visualizing Sequences

Let's remember back to our first arithmetic sequence $2,\ 4,\ 6,\ 8,\ ...$ If we were to make a table of values for this sequence, it would look something like what is shown below on the left.

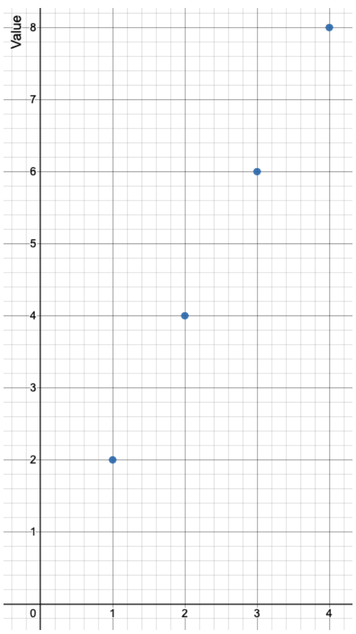
n	Value
1	2
2	4
3	6
4	8

On the left side of the table, we have all of our term numbers. The first term (when $n = 1$) has a value of 2. The next term (when $n = 2$) has a value of 4, then 6 when $n = 3$, and so on.

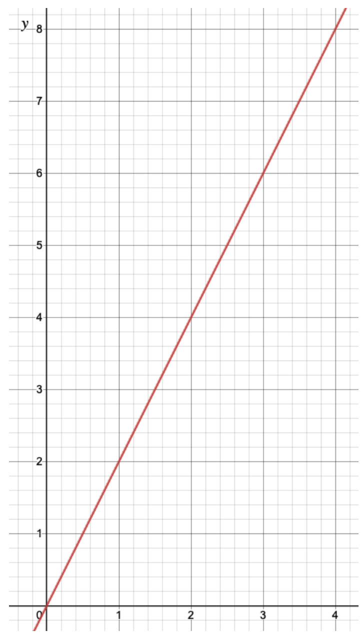
We can plot each point from the table of values on a graph. Our first term has a value of 2 when $n = 1$, then a value of 4 when $n = 2$, a value of 6 when $n = 3$ and a value of 8 when $n = 4$. Since every time n increases by 1 we increase the term value by 2, the value of any term will just be two times the term number, or $Value = 2 \cdot n$. So the 5^{th} term in the sequence will have a value of $2 \cdot 5 = 10$, etc. Notice we can fit a solid line through all of these points, as shown on the next page.

An equation representing this pattern does the same thing, except n can be any value it wants — it does not have to be a whole number like 1, 2 or 3. This is called the input x, which spits out the output y. So instead of $Value = 2 \cdot n$, the equation we graph would be $y = 2 \cdot x$. Plotting our points in Desmos, along with the equation $y = 2x$, we see that they both cross the exact same points and form the shape of a line. The only difference is that the equation $y = 2x$ can have any input value it wants, not just whole numbers like 1, 2 or 3. This is why it forms a line instead of points.

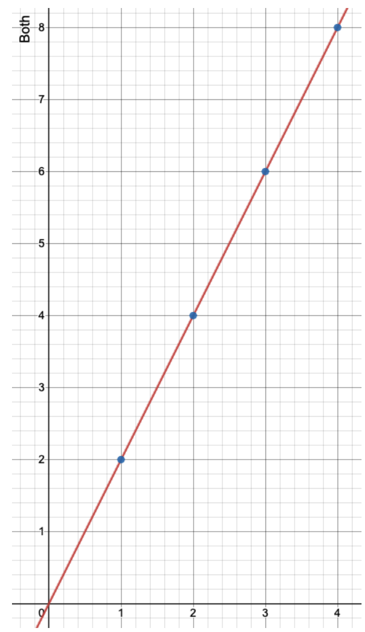
Sequence



Function



Sequence & Function



Arithmetic Sequence Formula

Now that we've covered the basics of an arithmetic sequence, let's use what we know to solve more interesting questions. What if we wanted to find the hundredth term in an arithmetic sequence?

Arithmetic Sequence Formula

For a sequence

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d+\ldots$$

where a is the starting value and d is the common difference, we can find any term in the sequence by using the formula

$$b = a + d \cdot (n-1)$$

where n is the term number we are interested in, and b is the value of the term at this position of the sequence; we call it the 'n^{th}' term.

Example 2

Consider the sequence

$$-1,\ 5,\ 11,\ \ldots$$

Could it be arithmetic? If so, find its 12^{th} term.

Solution:

The sequence appears arithmetic since each time we jump to the right, the value of each term increases by the same value: 6. Therefore $d = 6$, and we see the first term is $a = -1$. Also, since we want the 12^{th} term, we know $n = 12$. Plugging this into our formula, we can find the 12^{th} term, b.

$$b = a + d \cdot (n - 1)$$

$$b = -1 + 6 \cdot (12 - 1)$$

$$b = -1 + 6 \cdot 11 = -1 + 66 = 65$$

$$\therefore b = 65$$

Finding Sums

Until now, we have looked at sequences as collections of individual numbers. We were able to identify the starting term, the common difference, and could use the Arithmetic Sequence Formula to find any term in the sequence. However, we can *still* do more with these sequences.

Imagine we had the sequence $47,\ 51,\ 55,\ 59,\ 63$ representing the amount of money that five friends had each collected for a fundraising event. The first friend collected \$47, the second friend collected \$51, and so on. How could we find the total amount of money that the friends collected together?

Well, we could grab our calculators and plug in $47 + 51 + 55 + 59 + 63 = 275$. But what if instead of five friends, there were fifty? Using a calculator is no longer very fun. Let's notice that the friends' savings create an arithmetic sequence. If we want to add (or 'sum') the values of each term in the sequence, we can start by realising that every arithmetic sequence has an average term value. For the sequence above, the middle term, 55, will tell us the average value of the other numbers. Notice, the two numbers on either side of the middle number, 51 and 59, have an average of 55. Similarly, the average of the next two numbers 47 and 63 is also 55. This is because whether we move to the left or the right of this middle number, we are moving by 4 units each jump to the left or right.

Average Value of an Arithmetic Sequence

Sometimes it can be difficult to find the exact middle value of an arithmetic sequence, so we can use an easy trick. Since the common difference is always the same, we can find the average value of the sequence

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots\ b$$

by ignoring all of the inside terms and just finding the average of a and b using the formula

$$average\ = \frac{a+b}{2}$$

In the example above, the first and last values are $a = 47$ and $b = 63$. So the average value of the sequence is

$$average\ = \frac{a+b}{2} = \frac{47+63}{2} = \frac{110}{2} = 55$$

This average value is useful because if we want to find the total money fundraised, we can act as if each friend made exactly \$55. Even though we know this isn't true, the friends that made more or less money than \$55 will both equally pull each other towards this value of \$55.

Exercise 2

What is the average value of the arithmetic sequence

$$-61, -50,\ -39\ \ldots\ 71,\ 82,\ 93\ ?$$

Summing Arithmetic Sequences

So, the total amount of money the friends raised is just the number of friends, n, times how much money each friend makes (or the average between them). So the total sum of money is:

$$S = n \cdot average\ = n \cdot \left(\frac{a+b}{2}\right) = 5 \cdot 55 = 275$$

This is the same result as before, just much quicker!

We found above that $b = a + d \cdot (n-1)$. Plugging this back into our formula for the sum S, we get

$$S = n \cdot \left(\frac{a+b}{2}\right) = n \cdot \left(\frac{a + (a + d \cdot (n-1))}{2}\right) = n \cdot \left(\frac{2a + d \cdot (n-1)}{2}\right)$$

This leads us to perhaps our most useful result: a summation formula for arithmetic sequences.

Sum of Arithmetic Sequence Formula

The total sum S of n terms in an arithmetic sequence is

$$S = n \cdot \left(\frac{a+b}{2}\right) = \frac{n}{2}\left(2a + d \cdot (n-1)\right)$$

where a is the starting term, b is the final term, and d is the common difference.

We can use either formula depending on the information we know. If we know the value of the final term b, it is typically easier to use the formula $S = n \cdot \left(\frac{a+b}{2}\right)$. However, if we only know the first term a, it is best to use the formula $S = \frac{n}{2}\left(2a + d \cdot (n-1)\right)$.

Example 3

Determine the sum of the arithmetic sequence

$$-7,\ 13,\ 33,\ 53,\ 73,\ 93,\ 113$$

Which formula is easiest to use? Why?

Solution:

The difference between each term is $d = 13 - (-7) = 20$. The first term is $a = -7$, and we know there are exactly 7 terms, so $n = 7$. In this problem, we also know the last term $b = 113$. We then know a, b and n, so it is easiest to use the first summation formula $S = n \cdot \left(\frac{a+b}{2}\right)$ since knowing both a and b allows us to ignore all of the middle terms. Therefore, knowing the difference between each term is not necessary. Plugging these values into our formula, we have the sum

$$S = n \cdot \left(\frac{a+b}{2}\right)$$

$$= 7 \cdot \left(\frac{-7+113}{2}\right)$$

$$\therefore S = 7 \cdot \left(\frac{106}{2}\right) = 7 \cdot 53 = 371$$

So the total sum of the terms in the sequence is 371.

Solving for Different Variables

Finding the sum of an arithmetic sequence can be useful, but what would we do if we wanted to find other variables, like n?

Let's say we were given the sequence

$$-55,\ -40,\ -25,\ -10,\ ...\ 275,\ 290,\ 305,\ 320$$

How could we quickly find the number of terms in this sequence? Well, we first write down everything we know about the sequence. We know the first term is $a = -55$, the difference between each term is $d = -40 - (-55) = 15$, and the last term in the sequence is $b = 320$. If we remember back to our Arithmetic Sequence Formula, we found that

$$b = a + d \cdot (n - 1)$$

We know everything in this formula except for n, which is what we're trying find find. Perfect! Let's plug in our values. This will give us

$$320 = -55 + 15 \cdot (n - 1)$$

This is now an algebra problem. We are solving for n, so let's add 55 to both sides.

$$320 + 55 = \cancel{-55 + 55} + 15 \cdot (n - 1)$$

$$\implies 375 = 15 \cdot (n - 1)$$

We still just want n on the right side, so let's divide by 15 on both sides.

$$\frac{375}{15} = \frac{\cancel{15} \cdot (n - 1)}{\cancel{15}}$$

$$25 = n - 1$$

Now, all we need to do is add 1 to both sides so we just have n on the right side:

$$25 + 1 = n - \cancel{1} + \cancel{1}$$

$$\therefore n = 26$$

This means there are 26 terms in the sequence.

Example 4

Find the number of terms in the sequence below, then use this to find the sum of the sequence.

$$52,\ 91,\ \ldots\ 949,\ 988$$

Solution:

We see $a = 52$, $b = 988$, and $d = 91 - 52 = 39$. Plugging these into our Arithmetic Sequences Formula and solving, we get

$$b = a + d \cdot (n-1)$$

$$988 = 52 + 39 \cdot (n-1)$$

$$988 - 52 = \cancel{52} + 39 \cdot (n-1) - \cancel{52}$$

$$936 = 39 \cdot (n-1)$$

$$\frac{936}{39} = \frac{\cancel{39} \cdot (n-1)}{\cancel{39}}$$

$$24 = n - 1 \implies 24 + 1 = n - \cancel{1} + \cancel{1} = n$$

$$\therefore n = 25$$

For the sum of the sequence, we can use either formula from before. For simplicity, let's use the formula

$$S = n \cdot \left(\frac{a+b}{2}\right)$$

Plugging the values we know into the formula, we find that the sum is

$$S = n \cdot \left(\frac{a+b}{2}\right) = 25 \cdot \left(\frac{52+988}{2}\right) = 25 \cdot \left(\frac{1040}{2}\right) = 25 \cdot 520 = 13,000$$

So the total sum of the sequence is 13,000.

Exercise 3

Determine the value of n for the sequence below, then find its sum.

$$-33, -35, -37, \ ... \ -1021, \ -1023, \ -1025$$

Exercise 4

Is the sequence below arithmetic? How would you find its sum if it had many more terms?

$$1, \ 3, \ 9, \ 27, \ 81, \ 243$$